Geometric approach to the cryptanalysis of post-quantum multivariate signature schemes

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Ancient problem, modern solutions

Historical ciphers

- Polybus (150 B.C.)
- Caesar (50 B.C.)
- Vigenère (1586)

pen-and-paper ciphers

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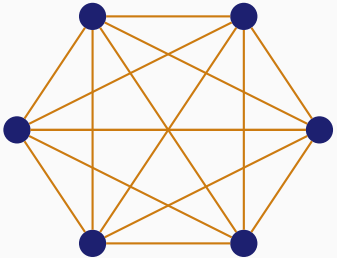
- DES/AES (1977/2000)

electromechanical/block ciphers

Secret key: one key per **link**:
6 participants: 15 keys.
100 participants: 4950 keys.



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- Diffie, Hellman (1973)
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$\longrightarrow$ secure internet, credit cards ...

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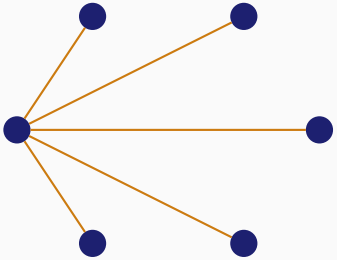

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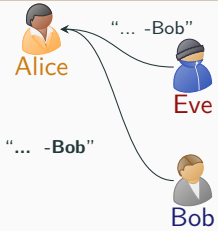
Modern cybersecurity goals

- **Confidentiality**: the information is only available to the intended recipient.
- **Integrity**: the information has not been modified after being sent.
- **Authenticity**: the information was sent by a specific, authenticated sender.

The public key revolution [Diffie, Hellman 1973]

Problem

- How to **authenticate** oneself over an **untrusted** network?
- How to verify **integrity** over an **untrusted** network?



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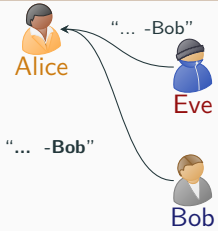

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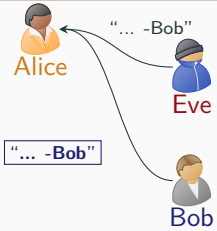

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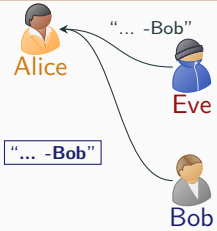

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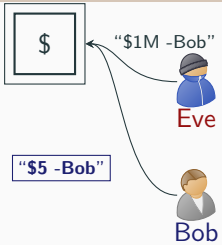

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Can **Eve** pretend to be **Bob**?

New revolution: Post Quantum Cryptography

Signatures in protocols

- TLS: create a **secure** channel over an untrusted network. 🔒 in your browser
- DNSSEC, SSH, ...

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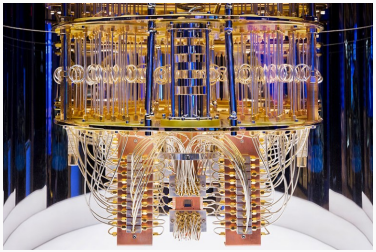

Source: IBM Research.

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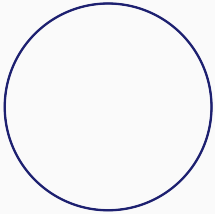
Algebraic geometry

The set of solutions of an **algebraic** equation is a **geometric** object called a **variety**.

$$X^2 + Y^2 - 1 = 0$$



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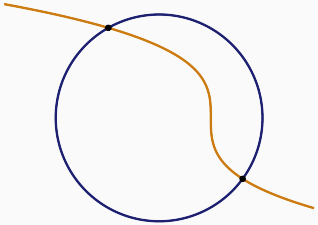

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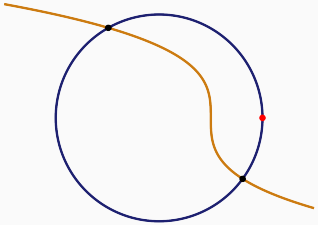

Polynomial system solving is NP

- Testing a solution is **easy**: "Is $(1, 0)$ a solution?"

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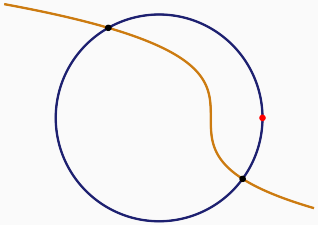

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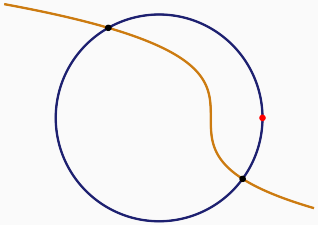

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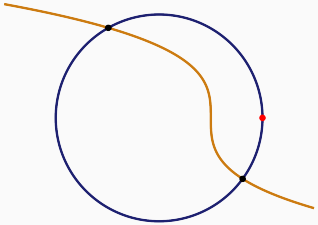

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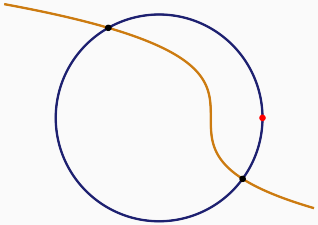

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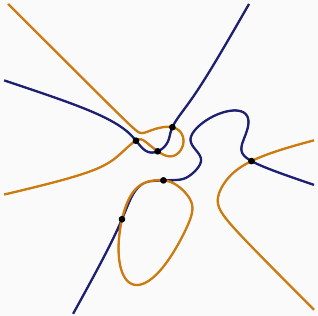

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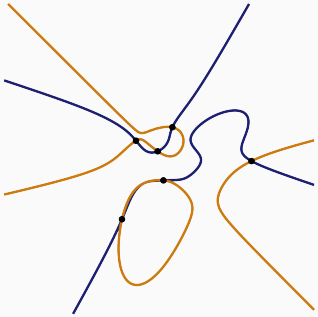

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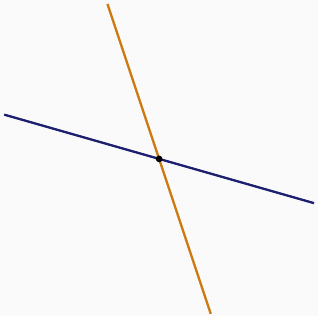

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Solving linear systems of equations is easy

Polynomial time algorithms for many linear algebra problems.

Multivariate signature schemes: a template

- **Public key**: a system of polynomial equations of degree two $\longrightarrow$ **Hard to solve**
- **Signature**: a solution of the public system of polynomial equations
- **Verification**: testing that a solution is correct $\longrightarrow$ **Easy to verify**

Unbalanced Oil and Vinegar [Patarin '97] [Kipnis, Patarin, Goubin '99]

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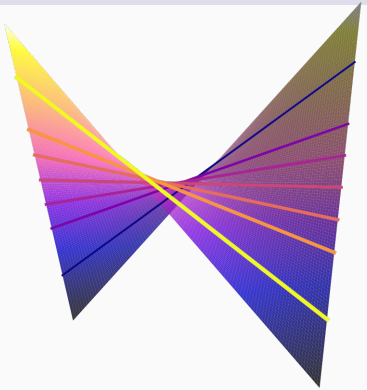

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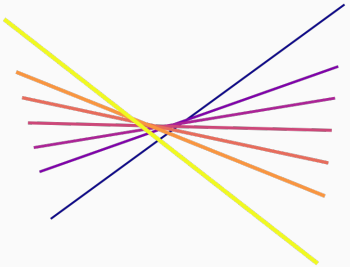

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Private key: polynomials $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ **linear** in $X_1, \ldots, X_o$, $A \in GL_n(\mathbb{F}_q)$.

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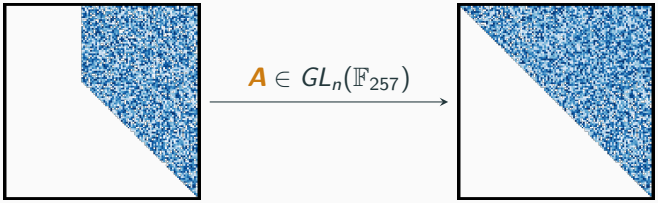

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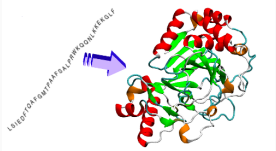

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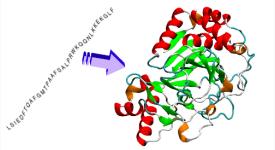

Mathematical history: [Descartes 1637], [Hilbert 1890], [Macaulay 1916], ...

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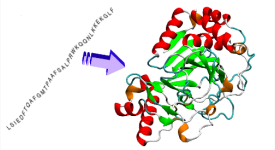

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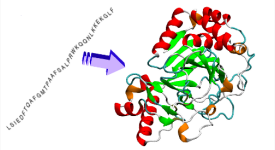

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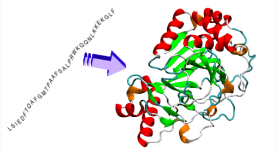

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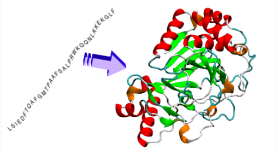

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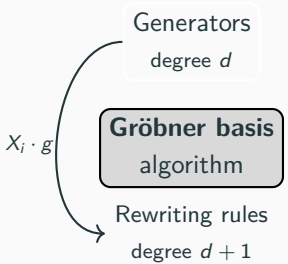

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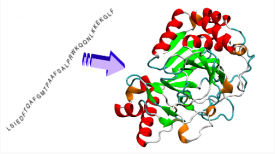

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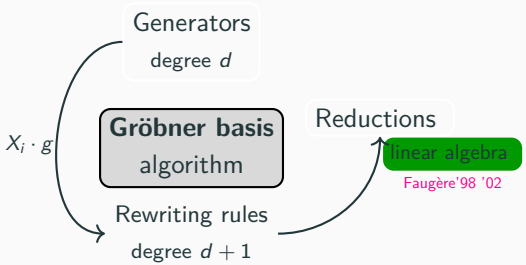

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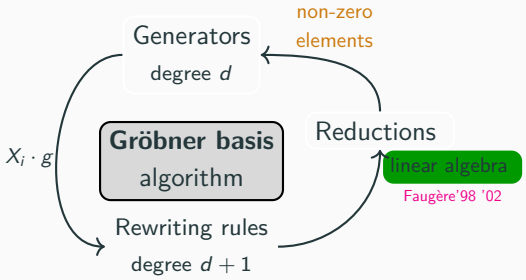

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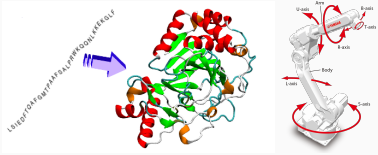

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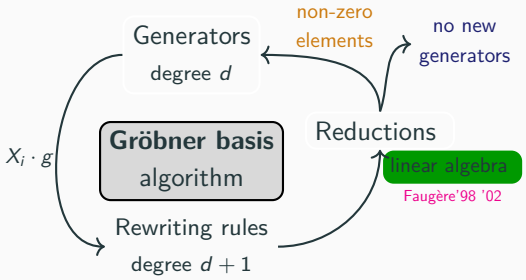

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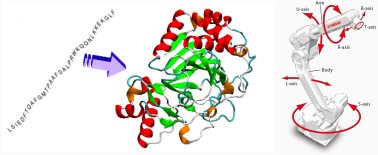

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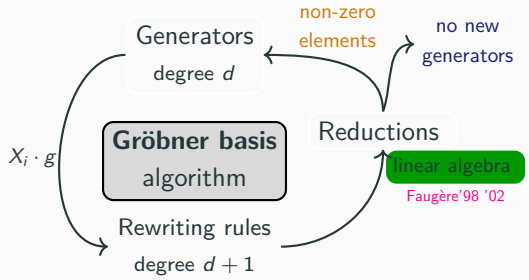

Direct attack

With Gröbner basis algorithms, public systems are **indistinguishable** from random systems.

n, m	UOV	Generic
$10, 4$	6,16,6	6,16,6
$12, 5$	7, 32, 7	7, 32, 7
$15, 6$	9, 64, 8	9, 64, 8
$17, 7$	10, 128, 9	10, 128, 8

Dimension, degree and degree of regularity for systems in n variables and m quadratic equations.

Exploiting the geometry of UOV

Original UOV formulation [Patarin 1997] [Kipnis, Patarin, Goubin 1999]

Private key: polynomials $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ **linear** in $X_1, \ldots, X_o$.

Public key: $p_1, \ldots, p_m \in \mathbb{F}_q[X_1, \ldots, X_n]$, $\mathcal{I} = \langle p_1, \ldots, p_m \rangle, A \in GL_n(\mathbb{F}_q)$, $p_i = f_i \circ A$.

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- $n > 2o$: $\mathcal{O}$ contains *eigenvectors* of some public matrices with probability $\approx q^{2o-n}$.
  Exponential-time attack, by computing $O(q^{n-2o})$ characteristic polynomials.

Tangent spaces of the UOV variety

Notations

Public key: polynomials $p_1, \ldots, p_m \in \mathbb{F}_q[X_1, \ldots, X_n], \mathcal{I} = \langle p_1, \ldots, p_m \rangle$ **radical**, $\operatorname{codim}\mathcal{I} = m$.
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Previous result: Reconciliation **[Ding, Yang, Chen, Chen, Cheng 2008]**

Given **one vector** $x \in \mathcal{O}$ and $\mathcal{P}$, compute a basis of $\mathcal{O}$ in time exponential in n, m.

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Input: a point $\boldsymbol{x} \in V$.

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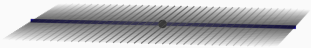

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Tangent space at a regular point

The tangent space of V at $\boldsymbol{x} \in V$ is

$$T_{\boldsymbol{x}} V := \ker(\mathrm{Jac}_{\mathcal{P}}(\boldsymbol{x}))$$

$$\mathrm{Jac}_{\mathcal{P}}(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial p_1}{\partial X_1} & \cdots & \frac{\partial p_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial p_m}{\partial X_1} & \cdots & \frac{\partial p_m}{\partial X_n} \end{pmatrix}$$

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Algorithm

Compute the restriction of $\mathcal{P}$ to $T_{\boldsymbol{x}} V$. Matrices have low rank if and only if $\boldsymbol{x} \in \mathcal{O}$.

Consequence: One vector to rule them all

Contribution: more than we bargained for **[P. PQC 2024]**

Given **one vector** $x \in \mathcal{O}$ and $\mathcal{P} = (p_1, \ldots, p_m)$, compute a basis of $\mathcal{O}$ in polynomial-time $O(mn^\omega)$, where $2 \leq \omega \leq 3$ is the exponent of matrix multiplication.

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Security level	I	I	III	V
n, m	112, 44	160, 64	184, 72	244, 96
Time	1.7s	4.4s	5.7s	13.3s
[DCCCY 08] (gates)	2^{55}	2^{65}	2^{75}	2^{92}
This work (gates)	2^{33}	2^{33}	2^{36}	2^{37}

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Complexity estimates and practical results with SageMath.

Experimental hardware: Intel i7 running at 2.80GHz with 8GB RAM.

Implementation available under MIT licence.

See also: [Aulbach, Campos, Krämer, Samardjiska, Stöttinger 2023]

Singular points of UOV and VOX

Limits of our previous result and locality of the UOV secret

- Not a full attack: require **side-channel** or **cryptanalysis** to obtain a vector $\boldsymbol{x} \in \mathcal{O}$.

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- YES, one obtains an **algebraic** variant of the **Kipnis-Shamir attack** [KPG '99].
- Attack does not break UOV, but applies succesfully to UOV$\hat{+}$/VOX.

Singular points and dimension of the tangent space

Tangent space and Jacobian matrix

Assume V equidimensional and $\mathcal{I}$ radical. The tangent space to V at $\boldsymbol{x}$ is the kernel of the Jacobian matrix evaluated at $\boldsymbol{x}$.

$$E : Y^2 - X^3 + 3X - 2 = 0.$$

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Previous work on geometric attacks

[KS'98] computes **singular points** of the intersection of two quadrics [Luyten 23]

[KPG'99] computes **singular points** of $V(\mathcal{I})$ Beullens, Castryck 23

Structured equations yield a structured Jacobian

Algebraic private key **[Kipnis, Patarin, Goubin 1999]**

Private key $\mathcal{F}$: m polynomials linear in $\boldsymbol{X_1}, \ldots, \boldsymbol{X_o}$, a linear map A: $1 \leq i \leq m, \quad p_i = f_i \circ A$.

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Main theorem: Dimension of the singular locus of $V(\mathcal{I})$ **[P. Eurocrypt 2025]**

UOV varieties admit a **positive dimensional** singular locus:

$$\dim \mathrm{Sing}(V(\mathcal{I})) \geq 2 \dim \mathcal{O} + m - n - 1$$

An algebraic attack targeting singular points

Generic smoothness outside of the secret subspace $\mathcal{O}$ **[P. Eurocrypt 2025]**

For a **generic** UOV variety, $\mathcal{I}$ is **radical** and $V(\mathcal{I})$ is **equidimensional of codimension** m.
In addition, $\mathrm{Sing}(V(\mathcal{I})) \subset \mathcal{O}$ (if the base field is large enough).

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Geometric interpretation of an old attack **[P. Eurocrypt 2025]**

[KS'98/KPG'99] are (hybrid) singular point attacks. Weaken hypotheses and support heuristic analysis by estimating $|\mathrm{Sing}(V(\mathcal{I}))|_{\mathbb{F}_q}$ with the Lang-Weil bound.

However, algebraic attack is more expensive than [KPG99] if q is "small".

Can the singularities be hidden to prevent attacks?

UOV$\hat{+}$ **[Faugère, Macario-Rat, Patarin, Perret 2022]**

In a UOV private key, replace $\boldsymbol{t} \leq \mathbf{8}$ polynomials by random quadratic polynomials, then **mix**.

$$\mathcal{P} = \mathcal{S} \circ \mathcal{F} \circ A, \quad \mathcal{I} = \langle p_1, \ldots, p_o \rangle.$$

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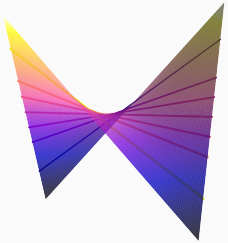
Geometric interpretation

$V(\mathcal{I})$ is the intersection of a UOV variety with t generic quadrics.

$$V(\mathcal{I}) = \qquad \cap \underbrace{V(\mathcal{J})}_{\text{UOV variety}}$$



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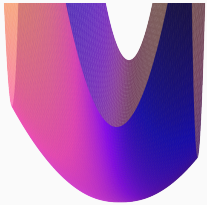

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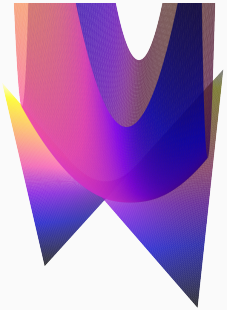

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Underlying UOV Jacobian

Jacobian of $\mathcal{F}$ when $\boldsymbol{x} \in \mathcal{O}$:

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Dimension computation [P. 2025]

$\hat{+}$ reduces the dimension of the singular locus by at most $\boldsymbol{2t}$ compared with UOV.

From singular points to a key recovery attack

$V(\mathcal{I})$ is the UOV$\hat{+}$ public key variety, $V(\mathcal{J})$ is the underlying UOV variety.

Singular points (still) leak the trapdoor

$$\mathrm{Sing}(V(\mathcal{I})) \subset \mathrm{Sing}(V(\mathcal{J})) \subset \mathcal{O}$$

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Can we decide "$\boldsymbol{x} \in \mathcal{O}$?" faster than $O(q^t n^{\omega})$?

Adapting "$x \in \mathcal{O}$?" to UOV$\hat{+}$ efficiently

Previous result for UOV [P. 2024]

Decide $x \in \mathcal{O}$? in polynomial time: $x \in \mathcal{O} \implies \mathcal{O} \subset T_x V$.

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Restricting to the tangent space $T_x V$

$\mathcal{P}_{|T_x V}(x)$ is an **easy** UOV$\hat{+}$ instance if $x \in \mathcal{O}$.

$\longrightarrow$ **Decide in polynomial time.**

New attack on UOV$\hat{+}$/VOX

[Patarin, Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud 2023]

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Decide $x \in \mathcal{O}$? using $O\left(\binom{n-o+2t-3}{4}^2 \binom{n-2o+2t+1}{2}\right) \subset O(n^{10})$ arithmetic operations.

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Previous result

This attack improves the Kipnis-Shamir attack which required:

$$O(q^{n-2o+2t} n^{\omega})$$

Practical results and bit complexity

Parameters	I	III	V
$\log_2$ gates	2^{39}	2^{41}	2^{43}
Time	2.3s	7.1s	16.4s

Figure 1: $\boldsymbol{x} \in \mathcal{O}$? with msolve[1] on UOV$\hat{+}$.

Intel i7 @ 2.80GHz, 8GB RAM.

[1] msolve.lip6.fr *Implementation under MIT licence*

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Cryptanalysis

- Initial UOV$\hat{+}$ parameter sets **do not meet NIST security requirement**.
- We propose alternative parameters for UOV$\hat{+}$ achieving between 27%-44% (expanded) public key size reduction compared with UOV.

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Can we do better?

Special case of UOV$\hat{+}$ called VOX with **additional structure** was submitted to NIST.

Can we exploit that structure to improve our attack?

Practical attack on VOX [Patarin, Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud 2023]

Dimension computation for VOX, seen as QR-UOV$\hat{+}(q, n, m, t, \ell)$

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Initial VOX parameters are insecure

MinRank attack [Furue, Ikematsu 2023]
Direct attack [P. 2024b]

Parameters	I	III	V
ℓ	6	7	8
ℓ'	6	7	8
Time	0.03s	0.11s	0.32s

Timing for the subfield attack on VOX, on Intel i7 @ 2.80GHz 8GB RAM with msolve.

Practical attack on VOX [Patarin, Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud 2023]

Dimension computation for VOX, seen as QR-UOV$\hat{+}(q, n, m, t, \ell)$

QR compression enables **big field attack**: UOV$\hat{+}(q^\ell, m/\ell, n/\ell, m, t)$ defines a **variety that contains $\mathcal{O} \cap V(\mathcal{G})$** but it should be the **empty variety** for a generic system.

Initial VOX parameters are insecure

MinRank attack [Furue, Ikematsu 2023]
Direct attack [P. 2024b]

Subfield attack [P. 2024b]

Alternative parameters exposed to direct attack through **subfields** $\mathbb{F}_{q^{\ell'}} \subset \mathbb{F}_{q^\ell}$.

Parameters	I	III	V
ℓ	6	7	8
ℓ'	6	7	8
Time	0.03s	0.11s	0.32s

Ic	IIIa	Vb
9	15	14
3	**5**	**7**
400h[1]	83.4s	12.4s

Timing for the subfield attack on VOX, on Intel i7 @ 2.80GHz 8GB RAM with msolve.

Implementation available under MIT licence.

[1] on Intel Xeon E7-4820, at 2.00GHz, peaking at 54.3GB RAM usage.

Future and on-going work - 1

Precomputations for undetermined systems

Task: Reduce $MQ(n, m)$ to $MQ(m - \mathbf{k}, m - \mathbf{k})$. For UOV, $k = 1$ in any field [Cheng, Hashimoto, Miura, Takagi 2014]. Can we go **beyond** $k = 1$?
$\longrightarrow$ *WIP*: Adapt work of Reid for $k = 2$.

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Generalize genericity results to all UOV variants.
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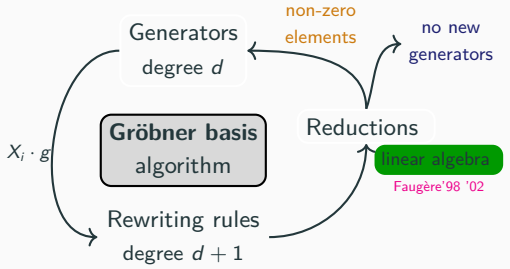

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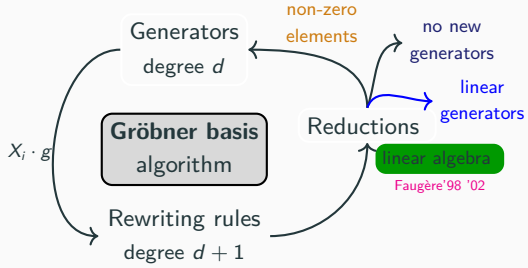

Future and on-going work - 2

Cryptanalysis of MAYO [Beullens 21]

Small dimension of secret subspace defeats UOV attacks.

- Algebraic structure induced by MAYO's whipped-up transform?
- Exploit **large** pre-image of signatures in EUF-CMA attacks?

	Signature (bytes)	Public key (bytes)
UOV-Ip	128	43 576
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Figure 3: Sizes at security level one.

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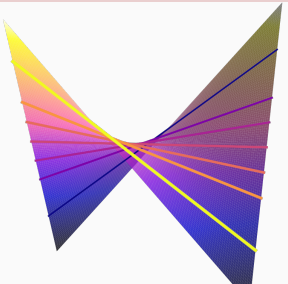

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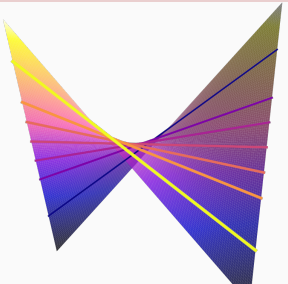

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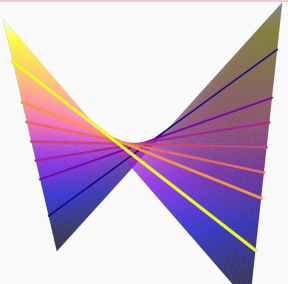
$\longrightarrow$ Dimension, degree and computational cost understood for a fixed projection.



Proposed UOV$\hat{+}$ parameters

Level	q, o, v, t	epk gain vs UOV
I	251, 48, 55, 6	36%
III	1021, 70, 79, 7	44%
V	4093, 96, 107, 8	27%

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Application to UOV

If $\alpha = \frac{n}{s}$ is a constant, then a UOV secret is characterized by a constant number of polynomials from the public key.
For practical parameters, 3 or 4 polynomials are enough.

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